Worcester County Mathematics League

Freshman/JV Meet 1
October 26, 2016

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Round 4 question 2
answer should be
210

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016

Round 1: Evaluation of Algebraic Expressions and Order of Operations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Simplify: $2(2-4[4+2\times 5]+40)+2\times 4$

2. If $a\otimes b=\frac{ab^2+(ab)^2}{b}$ and $a\odot b=a\div b+2\times a-1$, evaluate $3\otimes(1\odot\frac{1}{4})$.

3. For a function f, we can write $f(f(x))$ as $f^2(x)$. If $g(x)=10-\frac{x}{10}$, evaluate $g^6(10)$.

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________

(3 pts.) 3. ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016
Round 2: Solving Linear Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve for x: $5-3(x-3)=3-2x$

2. The sum of five consecutive odd numbers is 565. What is the largest of these five numbers?

3. Solve: $\frac{1}{2}x-\frac{2}{3}x+\frac{4}{5}x-\frac{6}{7}x=1-4x$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016
Round 3: Logic Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If 1 and 3 are the first two odd numbers, what is the 200th odd number?

2. In digital multiplication, answers are single digits and are obtained by writing only the units digit from a product in standard multiplication. If $\odot$ is the symbol for digital multiplication, compute: $17 \odot 199 \odot 29 \odot 53 \odot 241 \odot 65 \odot 389$.

3. If a = 9 and g = 5, and if each letter represents a unique digit from 0 to 9, what is the value of the word "bingo" if the following equation is true. (Note: the letter "o" need not represent the digit 0)

```
   a t o m
 + b o m b
 b i n g o
```

ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. ________________

(3 pts.) 3. bingo= ____________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016
Round 4: Ratios, Proportions, and Variation

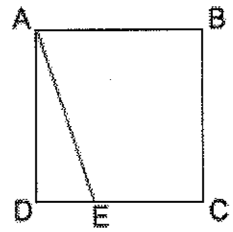
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If z varies directly with x, and $z = -5$ when $x = \frac{1}{4}$, find x when $z = -7$.

2. Farmer Greg has four different size sacks of flour (small, medium, large, and jumbo) and their total weight is 480 pounds. The ratio of the weight of the small sack to the medium sack is 5:12. The ratio of the weight of the medium sack to the large sack is 3:7. The ratio of the weight of the large sack to the jumbo sack is 4:5. How heavy is the jumbo sack of flour?

3. Suppose that ABCD is a square where AB = 27. Point E lies on side DC such that DE = ½ EC. If the ratio of AE to AD is k to 3, find k.



ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________ pounds

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

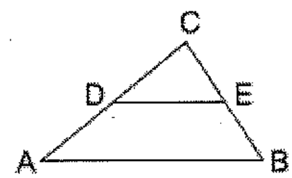
APPROVED CALCULATORS ALLOWED

1. Suppose $a \Delta b = ab - a^2$ and $a^{\Diamond} = \frac{-1}{a}$. Evaluate: $\left(-5 \Delta (-2)^{\Diamond}\right)^{\Diamond}$

2. Solve for x: $\frac{12x-6}{3} + 5(x-7) = -5(2x-15) + 11x$

3. In a logic puzzle, you are permitted to use the digits 0, 1, 3, 4, 5, 6, 7, and 8 only once each to create two four-digit numbers, A and B (Note: numbers cannot begin with 0). What is the smallest possible positive value of A - B?

4. In the figure, it is a fact that $\frac{DE}{AB} = \frac{CE}{CB}$. Suppose that DE = 45, AB = 72 and EB = 21. Find CE.



5. Factor completely: $4a^5b - 24a^4b^2 - 64a^3b^3$

6. If $3^x = 5$, find the value of 3^{2x+3}.

7. Solve for p: $(3p-5)^2 - 2(3p-5) + 1 = 0$

8. Sarah walked for awhile at 4 mph and then biked for the rest of the day at 20 mph. If she covered a total of 120 miles in 10 hours, how many miles did she walk?

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014
Team Round Answer Sheet

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________ miles

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016 ANSWER KEY

Round 1:

1. -20 (Auburn)
2. 60 (South)
3. 9.09091 (QSC)

Round 2:

1. 11 (Auburn)
2. 117 (WMass ARML)
3. $\frac{210}{793}$ (Quaboag)

Round 3:

1. 399 (Algonquin)
2. 5 (St. Peter-Marian)
3. 10753 (Tahanto)

Round 4:

1. $\frac{7}{20}$ or 0.35 (Doherty)
2. ~~175~~ 210 (Shrewsbury)
3. $\sqrt{10}$ (Bromfield)

TEAM Round

1. $\frac{2}{55}$ or $0.0\overline{36}$ (Groton)
2. 14 (Millbury)
3. 137 (Douglas)
4. 35 (Nashoba)
5. $4a^3b(a-8b)(a+2b)$ (Shepherd Hill)
6. 675 (Westboro)
7. 2 (Shrewsbury)
8. 20 (Millbury)

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 26, 2016 - SOLUTIONS

Round 1: Evaluation of Algebraic Expressions and Order of Operations

1. Simplify: $2(2-4[4+2\times 5]+40)+2\times 4$

Solution: Follow the order of operations:

$$2(2-4[4+2\times 5]+40)+2\times 4$$
$$2(2-4[4+10]+40)+2\times 4$$
$$2(2-4[14]+40)+2\times 4$$
$$2(2-56+40)+2\times 4$$
$$2(-14)+2\times 4$$
$$-28+2\times 4$$
$$-28+8=-20.$$

2. If $a\otimes b=\frac{ab^2+(ab)^2}{b}$ and $a\odot b=a\div b+2\times a-1$, evaluate $3\otimes(1\odot\frac{1}{4})$.

Solution 1 (Straight Evaluation): Begin with the expression inside the parentheses:

$$1\odot\tfrac{1}{4}$$
$$1\div\tfrac{1}{4}+2\times 1-1$$
$$1\times 4+2\times 1-1$$
$$4+2-1=5$$

Now we evaluate the second part of the expresion:

$$3\otimes 5$$
$$\frac{3(5)^2+(3\times 5)^2}{5}$$
$$\frac{3\times 25+15^2}{5}$$
$$\frac{75+225}{5}$$
$$\frac{300}{5}=60.$$

Solution 2 (Simplify Function First): Perform the same first step as in solution 1 to determine the value of $1\odot\frac{1}{4}$. Now note that

$$\frac{ab^2+(ab)^2}{b}=\frac{ab^2+a^2b^2}{b}=ab+a^2b=ab(1+a).$$

This means that $3\otimes 5=3\times 5\times(1+3)=15\times 4=60$.

3. For a function f, we can write $f(f(x))$ as $f^2(x)$. If $g(x) = 10 - \frac{x}{10}$, evaluate $g^6(10)$.

Solution 1 (Straight Computation): Begin by computing $g(x)$ and then move on to higher powers of the function.

$$g(10) = 10 - \tfrac{10}{10} = 9$$
$$g^2(10) = g(g(x)) = g(9) = 10 - \frac{9}{10} = 10 - 0.9 = 9.1$$
$$g^3(10) = g(g^2(10)) = g(9.1) = 10 - \frac{9.1}{10} = 10 - 0.91 = 9.09$$
$$g^4(10) = g(g^3(10)) = g(9.09) = 10 - \frac{9.09}{10} = 10 - 0.909 = 9.091$$
$$g^5(10) = g(g^4(10)) = g(9.1) = 10 - \frac{9.091}{10} = 10 - 0.9091 = 9.0909$$
$$g^6(10) = g(g^5(10)) = g(9.0909) = 10 - \frac{9.0909}{10} = 10 - 0.90909 = 9.09091$$

Solution 2 (Find Generalized Expression): Instead of computing the answer directly, attempt to find an expression for $g^n(x)$ where n is any integer.

$$g(x) = 10 - \tfrac{x}{10}$$
$$g^2(x) = g(g(x)) = g(10 - \tfrac{x}{10}) = g(9) = 10 - \frac{10 - \frac{x}{10}}{10} = 10 - 1 + \tfrac{x}{100}$$
$$g^3(x) = g(g^2(x)) = g(10 - 1 + \tfrac{x}{100}) = 10 - \frac{10 - 1 + \frac{x}{100}}{10} = 10 - 1 + 0.1 - \tfrac{x}{1000}$$
$$g^4(x) = g(g^3(x)) = g(10 - 1 + .1 - \tfrac{x}{1000}) = 10 - \frac{10 - 1 + .1 - \frac{x}{1000}}{10} = 10 - 1 + 0.1 - 0.01 + \tfrac{x}{10000}$$

At this point, a pattern is becoming clear. We have that:

$$g^n(x) = 10^1 - 10^0 + 10^{-1} + \ldots + (-1)^{n-1}10^{-(n-2)} + (-1)^n (10)^{-n} x$$

Therefore, we have that $g^6(10) = 10 - 1 + .1 - .01 + .001 - .0001 + 10 \times 10^{-6} = 9.09091$

Round 2: Solving Linear Equations

1. Solve for x: $5 - 3(x - 3) = 3 - 2x$

Solution: Simplify and then isolate x on one side of the equation:

$$5 - 3(x - 3) = 3 - 2x$$
$$5 - 3x + 9 = 3 - 2x$$
$$5 + 9 - 3 = -2x + 3x$$
$$11 = x$$

2. The sum of five consecutive odd numbers is 565. What is the largest of these five numbers?

Solution: Let x be the largest of the five odd numbers. We then have the following equation:

$$x+(x-2)+(x-4)+(x-6)+(x-8)=565$$
$$5x-20=565$$
$$5x=585$$
$$x=117$$

3. Solve: $\frac{1}{2}x-\frac{2}{3}x+\frac{4}{5}x-\frac{6}{7}x = 1-4x$

Solution: Begin by noting that the least common multiple of 2, 3, 5, and 7 is $2\times3\times5\times7=210$. Multiply both sides by 210 to get

$$210\left(\tfrac{1}{2}x-\tfrac{2}{3}x+\tfrac{4}{5}x-\tfrac{6}{7}x\right) = 210(1-4x)$$
$$105x-140x+168x-180x = 210-840x$$
$$-47x+840x = 210$$
$$793x = 210$$
$$x = \tfrac{210}{793}.$$

(Note: we know this answer is irreducible since 793 is not divisible by 2, 3, 5, or 7)

Round 3: Logic Problems

1. If 1 and 3 are the first two odd numbers, what is the 200th odd number?

Solution: We know that odd numbers have the form $2n+1$ for some n. For example, the first odd number is 1, which can be represented using $n=0$, since $2(0)+1=1$. If we want to find the 200th odd number, we simply need to plug in $n=199$. This gives $2(199)+1=398+1=399$.

2. In digital multiplication, answers are single digits and are obtained by writing only the units digit from a product in standard multiplication. If $\odot$ is the symbol for digital multiplication, compute: $17 \odot 199 \odot 29 \odot 53 \odot 241 \odot 65 \odot 389$.

Solution 1 (Full computation): Consider two integers a and b and let x be the units digit of a and y be the units digit of b. Notice that $a \odot b = x \odot y$. This is true since the units digit of a product only depends on the units digits of the two factors. Therefore, we can simplify the expression above to be

$$7 \odot 9 \odot 9 \odot 3 \odot 1 \odot 5 \odot 9$$

Now simply compute from left to right:

$7 \odot 9 = 3$ since $7 \times 9 = 63$.

$3 \odot 9 = 7$ since $3 \times 9 = 27$.

$7 \odot 3 = 1$ since $7 \times 3 = 21$.

$1 \odot 1 = 1$ since $1 \times 1 = 1$.

$1 \odot 5 = 5$ since $1 \times 5 = 5$.

$5 \odot 9 = 5$ since $5 \times 9 = 45$.

Therefore, the final answer is 5.

Solution 2: Use the same fact as in solution 1 to reduce the problem to $7 \odot 9 \odot 9 \odot 3 \odot 1 \odot 5 \odot 9$. Next, note that when an odd number is multiplied by 5, the units digit is always 5. Since every factor in this digital multiplication is odd and there is one factor of 5, we can immediately conclude that the answer must be 5.

3. If a = 9 and g = 5, and if each letter represents a unique digit from 0 to 9, what is the value of the word "bingo" if the following equation is true. (Note: the letter "o" need not represent the digit 0)

```
   a t o m
 + b o m b
 b i n g o
```

Solution: Since numbers cannot begin with 0 we can immediately see that b = 1.

Since we know that a = 9 and b = 1, we have that i must be either 0 or 1. However, each letter represents a unique digit and we already know that b = 1, so we can conclude that
i = 0.

Next, we have two equations relating the letters o and m:

$$m + b = o \quad \rightarrow \quad m + 1 = o \qquad (1)$$

$$o + m = g \quad \rightarrow \quad o + m = 5 \text{ or } o + m = 15 \qquad (2)$$

Since we already know that a = 9, equation (1) tells us that m and o are consecutive digits. Since the remaining digits available are 2, 3, 4, 6, 7, and 8, this means that (m, o) must be equal to one of (2, 3), (3, 4), (6, 7), or (7, 8). However, equation (2) lets us eliminate (3, 4) and (6, 7) as potential options. Therefore, we have that (m, o) = (2, 3) or (m, o) = (7, 8).

Assume that (m, o) is equal to (7, 8). We can now plug in our known values into the original equation to see

$$\begin{array}{r} 9\ t\ 8\ 7 \\ +\ 1\ 8\ 7\ 1 \\ \hline 1\ 0\ n\ 5\ 8 \end{array}$$

This means that t plus 8 plus the carryover 1 must have a units digit of n. However, since we know there is no carry over into the thousands column, we conclude that t + 8 + 1 < 10. But there is no possible value of t that makes this inequality true. Therefore, we know that our assumption that (m, o) = (7, 8) must be wrong and instead it must be true that (m, o) = (2, 3). In that case, the original sum becomes

$$\begin{array}{r} 9\ t\ 3\ 2 \\ +\ 1\ 3\ 2\ 1 \\ \hline 1\ 0\ n\ 5\ 3 \end{array}$$

Therefore, we have that t + 3 = n. Since the only remaining digits left are 4, 6, 7, and 8, we can immediately see that the only possible value of (t, n) is (4, 7).

This means the value of bingo is given by 10753.

Round 4: Ratios, Proportions, and Variation

1. If z varies directly with x, and $z = -5$ when $x = \frac{1}{4}$, find x when $z = -7$.

Solution: From the statement of the problem we know that for some k, it must be true that $z = kx$. Plugging in the known information we can compute k:

$$z = kx$$
$$-5 = k \times \tfrac{1}{4}$$
$$k = -20.$$

Now that we know the value of k, we have that

$$-7 = -20x$$
$$x = \tfrac{7}{20}.$$

2. Farmer Greg has four different size sacks of flour (small, medium, large, and jumbo) and their total weight is 480 pounds. The ratio of the weight of the small sack to the medium sack is 5:12. The ratio of the weight of the medium sack to the large sack is 3:7. The ratio of the weight of the large sack to the jumbo sack is 4:5. How heavy is the jumbo sack of flour?

Solution: To solve the problem, we need to find a ratio which directly relates all four sack sizes.

We are given that the ratio of small to medium to 5 : 12. We are given that the ratio of medium to large is 3 : 7, which we can also write as 12 : 28. Lastly, we are given that the ratio of large to jumbo is 4:5, which we can also write as 28 : 35.

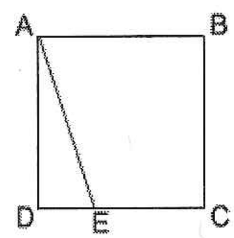
This means we can express the ratio of:

small to medium as 5 : 12
small to large as 5 : 28
small to jumbo as 5 : 35.

In total, we have that 5 + 12 + 28 + 35 = 80. This means that the weight of the jumbo sack of flour is given by $35 \times \frac{400}{80} = 35 \times 5 = 175$ pounds.

Should be $\frac{480}{80} = 35 \times 6 = 210$

3. Suppose that ABCD is a square where AB = 27. Point E lies on side DC such that DE = ½ EC. If the ratio of AE to AD is k to 3, find k.



Solution: Since ABCD is a square and AB = 27, we know that AD is also 27.

Next, note that DE + EC = DC = 27. This means that EC = 27 - DE. We are given that DE = ½ EC. Plugging in our expression for EC, we get

$$DE = \tfrac{1}{2} EC$$
$$DE = \tfrac{1}{2}(27 - DE)$$
$$2DE = 27 - DE$$
$$3DE = 27$$
$$DE = 9.$$

Since we now know AD and DE, we can compute AE using the Pythagorean formula.

$$AE = \sqrt{27^2 + 9^2}$$
$$AE = \sqrt{729 + 81}$$
$$AE = \sqrt{810} = \sqrt{9^2 \times 10} = 9\sqrt{10}.$$

From the given ratio, we know that

$$\frac{AE}{AD} = \frac{k}{3}$$
$$\frac{9\sqrt{10}}{27} = \frac{k}{3}$$
$$k = \sqrt{10}.$$

Team Round

1. Suppose $a \Delta b = ab - a^2$ and $a^\Diamond = \frac{-1}{a}$. Evaluate: $\left(-5 \Delta (-2)^\Diamond\right)^\Diamond$

Solution: We have that

$$\left(-5 \Delta (-2)^\Diamond\right)^\Diamond$$
$$\left(-5 \Delta \tfrac{1}{2}\right)^\Diamond$$
$$\left((-5) \times \tfrac{1}{2} - (-5)^2\right)^\Diamond$$
$$\left(\tfrac{-5}{2} - 25\right)^\Diamond$$
$$\left(\tfrac{-55}{2}\right)^\Diamond = \tfrac{2}{55}$$

2. Solve for x: $\frac{12x-6}{3} + 5(x-7) = -5(2x-15) + 11x$

Solution 1: Begin by multiplying both sides by 3 and then isolating x on one side.

$$\frac{12x-6}{3} + 5(x-7) = -5(2x-15) + 11x$$
$$12x - 6 + 15(x-7) = -15(2x-15) + 33x$$
$$12x - 6 + 15x - 105 = -30x + 225 + 33x$$
$$12x + 15x + 30x - 33x = 105 + 6 + 225$$
$$24x = 336$$
$$x = 14.$$

Solution 2: We have that

$$\frac{12x-6}{3} + 5(x-7) = -5(2x-15) + 11x$$
$$4x - 2 + 5(x-7) = -5(2x-15) + 11x$$
$$4x - 2 + 5x - 35 = -10x + 75 + 11x$$
$$9x - 37 = x + 75$$
$$8x = 112$$
$$x = 14.$$

3. In a logic puzzle, you are permitted to use the digits 0, 1, 3, 4, 5, 6, 7, and 8 only once each to create two four-digit numbers, A and B (Note: numbers cannot begin with 0). What is the smallest possible positive value of A - B?

Solution: To begin, notice that the thousands digit of A must be exactly one larger than the thousands digit of B. If this condition is not met, we will have that either $A - B \geq 1000$ or that $A - B < 0$.

There are only five ways this condition can be met. If we let x be the thousands digit of A and y be the thousands digit of B, we have that (x, y) can be one of the following:

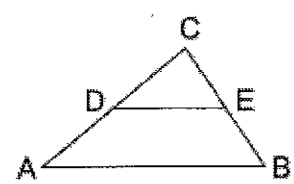
(4, 3), (5, 4), (6,5), (7,6), or (8,7).

For each of these cases, we will use the remaining digits to make A as small as possible and B as large as possible in order to minimize A - B.

	0, 1, 3, 4, 5, 6, 7, 8	
(4, 3):	A = 4015, B = 3876	A - B = 139
(5, 4):	A = 5013, B = 4876	A - B = 137
(6, 5):	A = 6013, B = 5874	A - B = 139
(7, 6):	A = 7013, B = 6854	A - B = 159
(8, 7):	A = 8013, B = 7654	A - B = 359

We conclude that the smallest possible positive value of A - B must be 137.

4. In the figure, it is a fact that $\frac{DE}{AB} = \frac{CE}{CB}$. Suppose that DE = 45, AB = 72 and EB = 21. Find CE.



Solution: Let x be the length of CE. Then from the given information we have that

$$\frac{DE}{AB} = \frac{CE}{CB}$$

$$\frac{45}{72} = \frac{x}{x+21}$$

$$45(x+21) = 72x$$

$$45x + 945 = 72x$$

$$945 = 27x$$

$$x = 35.$$

5. Factor completely: $4a^5b - 24a^4b^2 - 64a^3b^3$

Solution: Begin by factoring out the common factor of $4a^3b$ to get

$$4a^5b - 24a^4b^2 - 64a^3b^3$$
$$4a^3b\left(a^2 - 6ab - 16b^2\right)$$
$$4a^3b(a-8b)(a+2b)$$

6. If $3^x = 5$, find the value of 3^{2x+3}.

Solution: We know from the rules of exponents that

$$3^{2x+3} = 3^{x+x+3} = 3^x \times 3^x \times 3^3 = 5 \times 5 \times 27 = 675.$$

7. Solve for p: $(3p-5)^2 - 2(3p-5) + 1 = 0$

Solution 1 (Foil and Solve): Simplify the equation and then factor.

$$(3p-5)^2 - 2(3p-5) + 1 = 0$$
$$9p^2 - 30p + 25 - 6p + 10 + 1 = 0$$
$$9p^2 - 36p + 36 = 0$$
$$p^2 - 4p + 4 = 0$$
$$(p-2)(p-2) = 0.$$

Therefore, we know that $p = 2$.

Solution 2 (Substitution): Let $x = 3p - 5$. We can substitute this expression into the original equation to get

$$(3p-5)^2 - 2(3p-5) + 1 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0$$

Therefore, we know that $x = 1$. This gives us that

$$x = 3p - 5$$
$$1 = 3p - 5$$
$$6 = 3p$$
$$p = 2.$$

8. Sarah walked for awhile at 4 mph and then biked for the rest of the day at 20 mph. If she covered a total of 120 miles in 10 hours, how many miles did she walk?

Solution: Let x be the number of hours that Sarah walked. This means that she rode her bike for $10-x$ hours. We have that

$$4 \times x + 20 \times (10 - x) = 120$$

$$4x + 200 - 20x = 120$$

$$80 = 16x$$

$$x = 5.$$

Since she walked for 5 hours at a rate of 4 mph, we know Sarah walked 20 miles.